The statistics of turbulence

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Summary

This work takes up the fluid mechanics scheme first proposed by Ludwig Prandtl, according to which a fluid in turbulent motion is represented by a sort of gas made up of many particles interacting inelastically. For such a set of quasi-particles it can be shown that the Maxwell-Boltzmann distribution holds, whether one considers only one component of the velocity, or the full three-dimensional case. However, when the incidence of viscosity grows, such classical statistical analysis must instead be replaced by Bose-Einstein quantum statistics, for which a universal quantisation constant, determined in the author's Hydraulics Laboratory, has been found to be valid for all Newtonian fluids. By applying such a quantum statistical approach to the limit layer, the spectrum of velocity oscillations can be determined, thereby providing an explanation of the formation of ripples on an initially flat bottom.

Classical and quantum statistics

In 1925 L. Prandtl first introduced the perfect gas model to explain the behaviour of a real fluid in turbulent motion (Prandtl 1952). In fact, at small scales, turbulence is constituted by vortices that can be likened to a set of particles. Initially, these maintain their own individuality, but subsequently yield the momentum by mixing with other surrounding particles. Therefore, we can apply statistical analysis to this population of quasi-particles. In classical statistics, what is considered is a specific parameter, which must be discretized by dividing its possible values into a finite number of intervals and finally arriving at a count of the elements found in each of these intervals. The disctretizing operation is however usually quite arbitrary, the size of the chosen range generally being dictated by practical considerations alone. This involves some risks: if too large an interval is adopted, resolution is lost, while if instead very small ranges are used, the drawback arises of finding very few elements and therefore obtaining very scattered values. Nevertheless, as will be seen, it is sometimes useful to consider infinitesimal intervals. If, on the other hand, the energy is quantized, the intervals are already defined, as has been seen for harmonic oscillators (Buffoni 1996), and the discretization becomes natural and no longer arbitrary, as in the foregoing case. At the limit state, when the intervals are exceedingly narrow (that is viscosity $\nu \to 0$), such treatment reverts to a classical statistical

approach. However, the manner in which the division is performed is not the only difference between the classical and quantum statistical approaches. The two approaches also differ in the procedure for counting the number of states assumed by the system (Buffoni 1999). By way of example, let us suppose there are two distinct elements, a and b, which may exist in only two states, corresponding to two energy levels, 1 and 2. (By state, we mean the distribution of the elements in the two different discrete intervals.) In such a case, we will have two elements in the first interval and none in the second; then two cases (by switching a and b) corresponding to one element in the first and one in the second, and finally none in the first state and both in the second. There are thus four possible ways to arrange the two particles and, therefore, four accessible states. Note that while the two particles, a and b, are distinguished one from the other, we are not concerned with the order in which they are arranged within the single state. Obviously, this presumes, at least in principle, that we are able to distinguish the individual elements. If this were not possible, then the number of states would clearly be reduced. In fact, in the preceding example we would have indistinguishable elements a and a , and the accessible states would, in effect, be only three (two elements in the first interval; one in the first and the other in the second, both elements in the second), as the two elements could not be differentiated one from the other. The first way of proceeding corresponds to Maxwell-Boltzmann statistics, while the latter is adopted in Bose-Einstein statistics.

Boltzmann statistics

Let us consider a large number of independent systems (oscillators or quasi-particles) in statistical equilibrium with a source that has mean energy T, in such way that n_1 is found on level ε_1, n_2 on ε_2 and n_i on ε_i. Such a distribution on discrete levels can be obtained in the case of quantized energy, but also through classical statistical procedures by dividing the total energy into a finite number of intervals, thereby obtaining the same result. Of the many distributions that the system may assume over the various energy levels, we aim to calculate the most probable, which will therefore be that occurring most frequently in nature. Specifically, given a system in equilibrium with a source of mean energy T, and able to assume the discrete energy levels:

$$\varepsilon_1, \varepsilon_2, \varepsilon_3...\varepsilon_i$$

ask what the probability is that one of them be found on any given level ε_i. Alternatively, the problem can be framed in an entirely equivalent method by considering a very large number of independent systems, all in equilibrium with a source with energy T, and searching for the most probable number to be found on any given level ε_i, It can easily be shown that the most probable number of elements n_i assuming energy level ε_i, is given precisely by Boltzmann statistics (Fermi, 1934):

$$n_i = Ae^{-\frac{\varepsilon_i}{T}}. \tag{1}$$

Maxwell distribution

Boltzmann statistics (1) can be used to determine the actual distribution of the kinetic energy of N particles contained in the volume corresponding to one unit of mass. In classical statistics there is no size limitation on the cells in the phase space. Therefore, let us consider the space of momentum that coincides with that of the velocities and divide it into infinitesimal sub-cells in such way that their number is equal to the volume. We consider the volume contained in two spheres of radius u and $u+du$ precisely because we are looking for the number of particles having a kinetic energy per unit mass between ε and $\varepsilon+d\varepsilon$. The relation between ε and u is:

$$u^2 = 2\varepsilon \tag{2}$$

and we therefore have:

$$du = \frac{\sqrt{2}}{2}\varepsilon^{-1/2}d\varepsilon. \tag{3}$$

The volume between two spheres in the velocity space will be:

$$4\pi u^2 du. \tag{4}$$

Therefore, the number of sub-cells becomes:

$$4\pi\sqrt{2}\sqrt{\varepsilon}d\varepsilon. \tag{5}$$

However, in every cell there are N particles, varying as a function of the energy according to Boltzmann statistics (1), for which their total number will be:

$$dN = 4\pi A\sqrt{2}\sqrt{\varepsilon}e^{-\frac{\varepsilon}{T}}d\varepsilon, \tag{6}$$

where $T = u^2$. As each particle possesses an energy equal to ε, we have the infinitesimal total energy interval dE:

$$dE = 4\pi A\sqrt{2}\varepsilon^{3/2}e^{-\frac{\varepsilon}{T}}d\varepsilon. \tag{7}$$

$dE/d\varepsilon$ represents precisely the distribution function, that is, the number of particles whose kinetic energy is within the interval ε to $\varepsilon + d\varepsilon$. Constant A remains to be calculated, and to this end we impose the normalization condition:

$$4\pi A\sqrt{2}\int_0^\infty \sqrt{\varepsilon}e^{-\frac{\varepsilon}{T}}d\varepsilon = 1. \tag{8}$$

By setting $x = \varepsilon/T$ and therefore $d\varepsilon = 2Txdx$, the integral of the preceding relations yields:

$$2T^{3/2}\int_0^\infty x^2 e^{-x^2}dx = \frac{\sqrt{\pi}}{2}T^{3/2} \tag{9}$$

and, in conclusion, by substituting this into (7) we obtain:

$$A = \left(\frac{1}{2\pi T}\right)^{3/2}. \tag{10}$$

Therefore, from (7) we have derived function F, that is, the proportion of the N particles contained in one mass unit that has a kinetic energy value between ε and $\varepsilon + d\varepsilon$:

$$F(\varepsilon, T) = \frac{2}{\sqrt{\pi}T^{3/2}}\varepsilon^{3/2}e^{-\frac{\varepsilon}{T}}. \tag{11}$$

As this distribution was first derived via a different procedure by Maxwell in 1859, it has come to be known as the Maxwell distribution. It is often expressed as a function of velocity. In fact, if, as by convention, we express the volume of the velocity space as:

$$4\pi u^2 du, \tag{12}$$

the number of the sub-cells contained in this volume will be:

$$dN = 4\pi A u^2 e^{-\frac{u^2}{2T}} du. \tag{13}$$

Once again, in order to calculate constant A, we apply the normalization condition:

$$4\pi A \int_0^\infty u^2 e^{-\frac{u^2}{2T}} du = 1, \tag{14}$$

which yields:

$$A = \frac{1}{(2\pi T)^{3/2}} \tag{15}$$

and the Maxwell distribution expressed in terms of velocity finally becomes:

$$F(u, T) = \frac{\sqrt{2}}{\sqrt{\pi} T^{3/2}} u^2 e^{-\frac{u^2}{2T}}. \tag{16}$$

The case of a single component of velocity

Available instruments often allow measuring only one velocity component at a time. It is therefore useful to consider the statistics of one component of the velocity in isolation. In this case (still referring to one mass unit), the space of the momentum reduces to the space of the velocities and, given the arbitrariness of the cells' dimensions, we can divide this space into sub-cells of infinitesimal size. The number of accessible states therefore comes to coincide with the volume u and the number of sub-cells will be simply du. According to Boltzmann statistics (1), each sub-cell contains a number of particles, dN, by which, in conclusion, we obtain:

$$dN = A e^{-\frac{u^2}{2T}} du. \tag{17}$$

We must still determine constant A via the normalization condition:

$$\int_{-\infty}^\infty A e^{-\frac{u^2}{2T}} du = 1, \tag{18}$$

from which it is a simple matter to arrive at:

$$A = \frac{1}{\sqrt{2\pi T}}. \tag{19}$$

Therefore, the fraction of N particles whose velocity component lies on the interval between u and $u + du$ will be:

$$n(u, T) = \frac{1}{\sqrt{2\pi T}} e^{-\frac{u^2}{2T}}. \tag{20}$$

As can be gathered, the foregoing represents a Gaussian distribution with variance $\sigma^2 = T$.

Quantum statistics

A Newtonian fluid at low Reynolds numbers have a quantum behaviour, as has been encountered in the separation of vortices under critical and subcritical conditions (Buffoni, 1996, 1997, 1999), for which the universal quantization constant, valid for all fluids, was determined to be $k_o = 2,550\nu$ (where ν indicates the kinematic viscosity). In this case, the quasi-particles are in principle indistinguishable, and Boltzmann statistics give way to Bose statistics (Fermi 1934):

$$n_i = \frac{Q_i}{Ae^{\frac{\varepsilon_i}{T}} - 1}, \tag{21}$$

which for $Ae^{\varepsilon/T} >> 1$, reduces once again to classical Boltzmann statistics. Q_i is none other than the number of the quantum states, though, due to the limitation on the cells in the a count can no longer be arrived at as in the classical case. In fact, their volume can no longer be considered infinitesimal, but equal to k_o for one dimension, and $k_o^3 3$ for the three-dimensional case. This leads to the Bose-Einstein distribution for a single velocity component:

$$n_E(u, T) = \frac{1}{k_o(Ae^{\frac{\varepsilon_i}{T}} - 1)}, \tag{22}$$

while in three dimensions, we have:

$$F_E(u, T) = \frac{4\pi}{k_o^3} \frac{u^2}{Ae^{\frac{\varepsilon_i}{T}} - 1}, \tag{23}$$

In both cases, constant A is determined via the normalization condition, and the corresponding integral can be calculated either numerically or through a powers series.

Experimental checks

With the aim of verifying the foregoing theoretical formulations, we used customary single-component LDA techniques to measure the turbulence in a standard magnetic agitator in which the composition of the fluid used was varied: pure water; a 50% mixture of glycerin and water; and finally pure glycerin maintained via an electronic control system at a constant temperature of 100C ± 1. The relative viscosity was determined with the Guzman-Andrade equation. The velocity data obtained thereby were processed statistically in order to compare them with the results of the foregoing theoretical formulations. The results for the most significant cases are shown in the figures: water (fig. 1), the 1:1 glycerin + water mixture (fig. 2) and pure glycerin maintained at 100C (fig. 3). The results from classical Maxwell-Boltzmann statistics (20) are represented by the dashed line, while those relative to Bose-Einstein (22) by the continuous line; finally, the gray line shows the relative experimental histogram. Each histogram also shows the mean velocity in the direction of flow, the mean quadratic deviation, the kinematic viscosity, constant A and, finally, the value of the integral of the Einstein distribution in order to highlight compliance to the normalization condition. For low-viscosity fluids, such as water, the classical Maxwell distribution clearly holds (fig. 1). As for the others (fig. 2, 3), the difference between the classical theory of Maxwell (20) and the Bose-Einstein quantum theory (22) is modest, the discrepancy being roughly 8Although variations in experimental data often preclude deciding which of the two is valid, it is nevertheless safe to say that the data does not contravene the quantum hypothesis.

The boundary layer spectrum

An important field of application of quantum statistics is viscous substrata (Buffoni 1999). Let us therefore consider quantum quasi-particles with energy $\varepsilon = v^{*2}$ a momentum $q = k_o f/v^*$, where v^* indicates the shear velocity. What we must determine is the number of energy quanta whose frequency values are in the range f to $f + df$. Applying the Bose

distribution (7) to the quntum gas of boundary layer we must bear in mind that, because the particles are generated and absorbed continually, their number is not conserved, and we therefore have $A = 1$. Hence, in conclusion, wed obtain the spectrum, that is, the energy density per unit time within in the interval df:

$$u(f, T) = \frac{8\pi}{\upsilon^{*3}} \frac{kf^3}{e^{\frac{kf}{T}} - 1},$$ (24)

which is Planck's formula, where however the universal quantization constant equals $k_o = 2,550\nu$ and holds for all Newtonian fluids (Buffoni 1999). The maximum value of (24) occurs in correspondence to the wavelength $\lambda = 904\nu/\upsilon^*$. By Newton's relation, such oscillations generate a wide spectrum of tangential actions and therefore an initially flat bottom of fine sand will first be deformed in small waves, though subsequently higher energy waves will prevail until the ripples formed exhibit a λ practically in conformity with the maximum value furnished by expression (24). This quantum mechanism of ripples formation was hypothesized by the author in the late seventies through a relation analogous to De Broglie's ($\lambda = const.\nu/\upsilon^*$). Subsequently, it came to light that Yalin had also arrived at this same relationship empirically, specifying a constant value of about 2,000 for the fully developed ripples, and half that (i.e. 1,000) for those in the initial stages (Yalin 1977). Further experimental studies performed at the Hydraulics Institute of Pisa (Buffoni 1985, 1989) confirmed these values for low Reynolds numbers. Therefore, the quantum hypothesis finds a natural application in explaining the origins of the ripples, for which it provides a simple model for determining the spectrum of boundary layer oscillations underlying their formation.

-+ Bibliography

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