On shear stresses in tapered beams

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Tapered beams are widely used in both civil and industrial engineering, where such elements allow for a more efficient distribution of material in comparison to prismatic beams [1–3]. Taper affects the stress distribution in a beam. In particular, the distribution of shear stresses may be completely different from that observed in prismatic beams [4]. Strange as it may seem, this issue is normally not taken into account by standard design procedures based on beam element models, with consequent possible under- or over-estimation of the stress states acting in a real structure.

The present work aims at shedding light on this question by studying the effects of taper on shear stresses in a purposely simplified problem. A symmetrically tapered beam of length L, constant width, b, and constant taper angle, Λ, is considered. The material is homogeneous, linearly elastic, and isotropic. The beam is clamped at one end and loaded at the other one by an axial force, $\bar{N}$, a shear force, $\bar{Q}$, and a bending moment, $\bar{M}$. Let us assume a reference system Oyz, with the origin at the clamped end, and the z-axis coincident with the beam centreline (Fig 1).

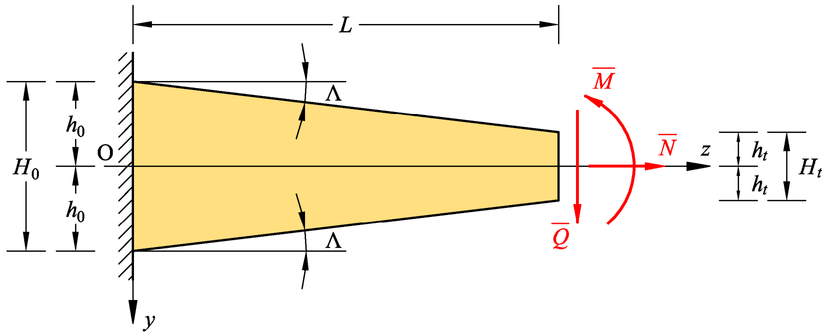

Figure 1: tapered cantilever beam subjected to tip loads.

Classical elasticity solutions related to the stated problem can be found in the literature. Michel and Carothers respectively determined the stresses in polar coordinates in an infinite elastic wedge loaded at its tip by a concentrated force [5] and a bending moment [6]. Galerkin [7] and Knops and Villaggio [8] deduced the solution for a truncated wedge by suitably superimposing the solutions for the wedge. Using those exact solutions as a starting point, we have determined the stress distribution in Cartesian coordinates in a tip-loaded elastic wedge, truncated or not. Subsequently, through the constitutive equations for a Lamé material, we have obtained the strain components and by integration we have determined also the displacement field. The complete analytical solution thus obtained shows many interesting features. In contrast to prismatic beams, shear stresses are produced not only by shearing forces, but also by pure extension and bending. Nevertheless, the final analytical expressions are cumbersome and unsuited for practical application. This has motivated the search for practical formulas through an approximate solution.

In Ref. [4], an extension of Jourawski's formula for tapered beams subjected to shear is obtained by imposing equilibrium of an infinitesimally short beam segment. Following the same approach, we have deduced the following expression for the (average) shear stress at y = const.:

$$\tau_{zy} = \frac{1}{b}\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{S_x^*}{I_x}M - \frac{A^*}{A}N\right), \tag{1}$$

where N and M respectively are the axial force and bending moment; A and I_x respectively denote the area and second moment of area of the cross section; A^* and S_x^* respectively are the area and first moment of area of the part of cross section above the ordinate y.

The exact and approximate analytical solutions have been compared with reference to a sample problem, obtaining a very good agreement in terms of shear stresses (Fig. 2). Very close results for the same problem have been obtained also via a finite element model using the commercial software Abaqus with plane stress shell elements (CPS8R) [9]. Extension of the analytical solution to general cross sections and implementation into a FEM framework are in progress.

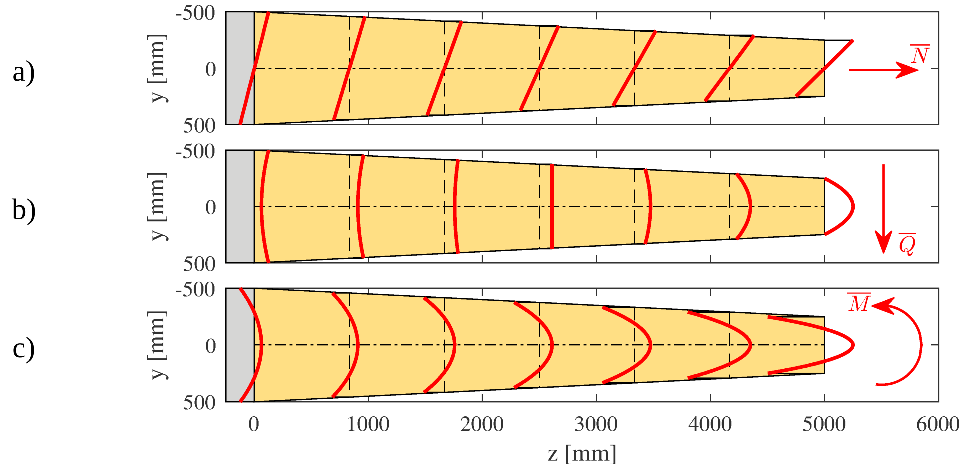

Figure 2: shear stresses under: a) axial force; b) shear force; c) bending moment.

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