Dynamics of a Swing

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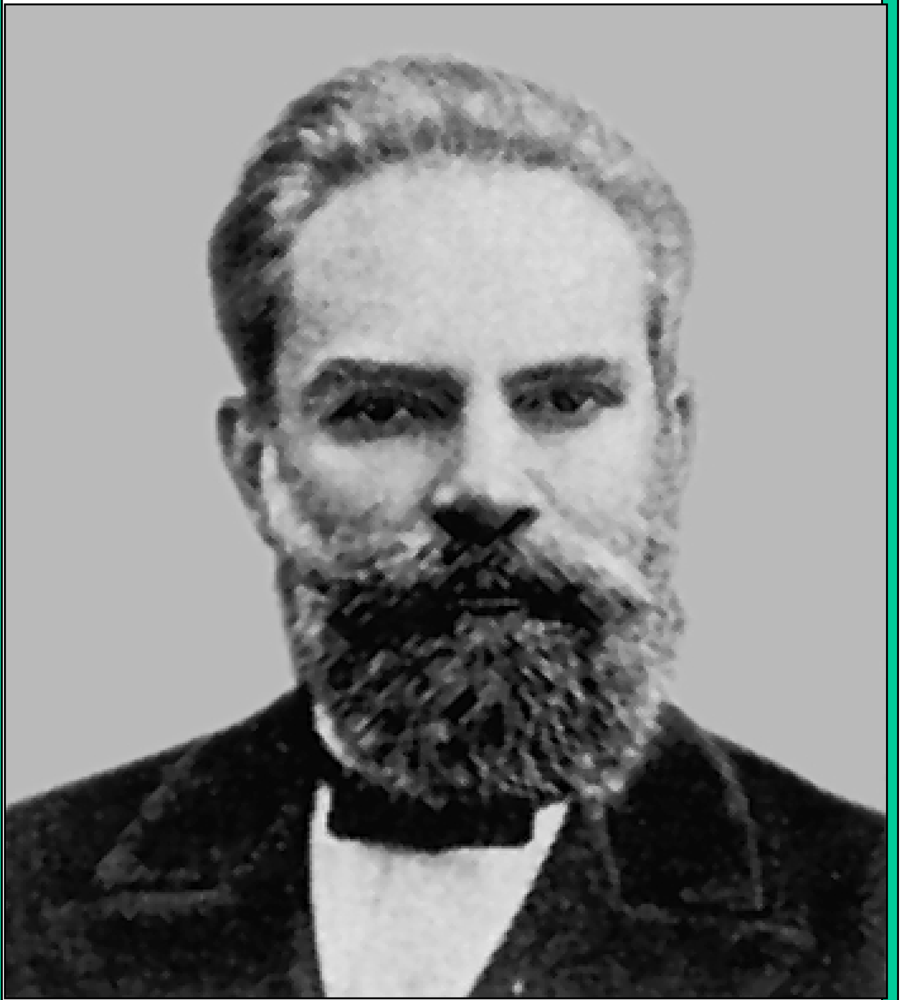
Key questions

- *Instability regions for the swing*
- *Regular and chaotic motions*

Methods

- *Stability analysis is based on derivatives of the Floquet matrix with respect to problem parameters*
- *Averaging method*
- *Numerical simulation*

Periodical Systems



150th Anniversary of
Alexander M. Lyapunov (1857 – 1918)

Thesis (1892)

«General problem on stability of motion»

Chapter 3

Study of periodic movements

$$\dot{\mathbf{x}} = \mathbf{G}\mathbf{x}$$

Introduction of parameters

$$\mathbf{G}(t,\mathbf{p}) = \mathbf{G}(t+T,\mathbf{p})$$

$\mathbf{p}$ – vector of parameters

Stability of periodic systems

General stability theory by Floquet (1883)

$$\dot{\mathbf{x}} = \mathbf{G}(t)\mathbf{x}\,, \quad \mathbf{G}(t) = \mathbf{G}(t+T)$$

matriciant

$$\dot{\mathbf{X}} = \mathbf{G}(t)\mathbf{X}, \quad \mathbf{X}(0) = \mathbf{I}$$

Monodromy matrix

$$\mathbf{F} = \mathbf{X}(T)$$

multipliers ρ

$$\mathbf{Fu} = \rho\,\mathbf{u}$$

Asymptotic stability

$$|\,\rho\,| < 1$$

Instability

$$|\,\rho\,| > 1$$

Bifurcations of multipliers

$$\frac{\partial \mathbf{F}}{\partial p_j} = \mathbf{F}\int_0^T \mathbf{X}^{-1}\frac{\partial \mathbf{G}}{\partial p_j}\mathbf{X}\,dt$$

$$\frac{\partial \rho}{\partial p_j} = \rho\,\mathbf{v}^T\int_0^T \mathbf{X}^{-1}\frac{\partial \mathbf{G}}{\partial p_j}\mathbf{X}\mathbf{u}\,dt$$

p_j – *a system parameter*

Seyranian, Solem, Pedersen (1999) Arch. Appl. Mech.

A swing – simplest model for parametric resonance

Nonlinear system -
a pendulum of variable length:

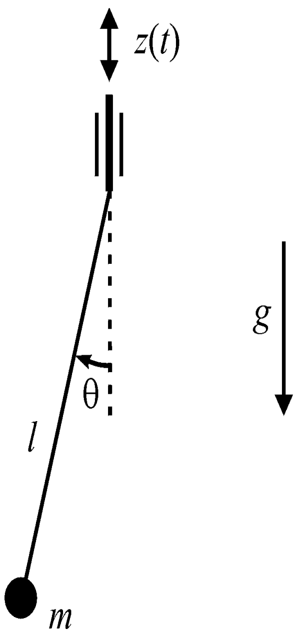
$$(ml^2\dot{\theta})^{\cdot} + \gamma l^2 \dot{\theta} + mgl\sin\theta = 0$$

$$l(t) = l_0 + a\varphi(\Omega t)$$

Resonant frequencies:

$$\Omega_k = \frac{2}{k}\sqrt{\frac{g}{l_0}},$$

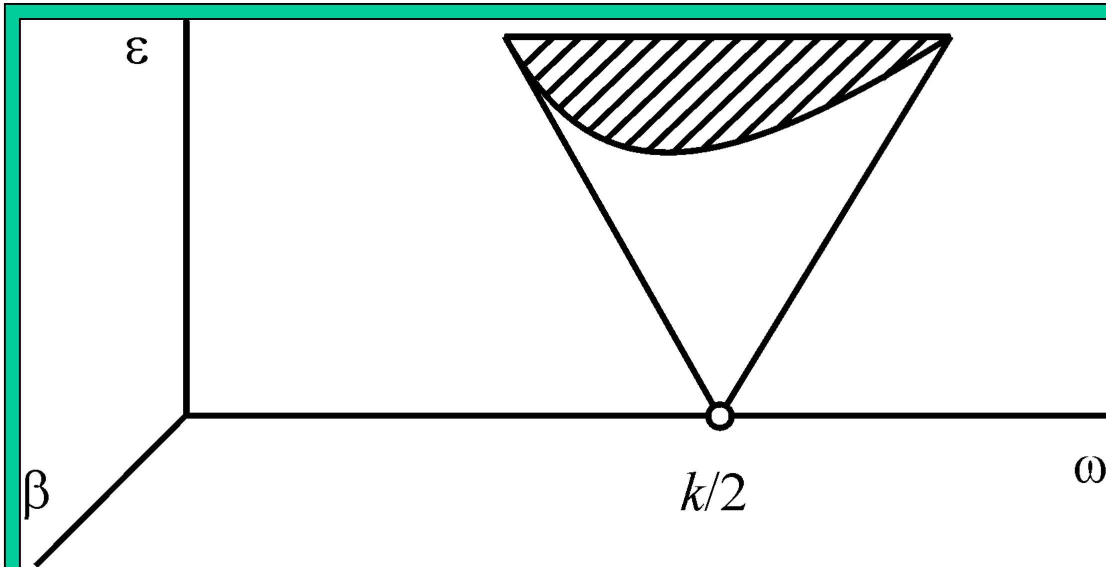
$$k = 1,2,\ldots$$



Three non-dimensional parameters:

$$\varepsilon = \frac{a}{l_0},\quad \omega = \frac{1}{\Omega}\sqrt{\frac{g}{l_0}},\quad \beta = \frac{\gamma}{m}\sqrt{\frac{l_0}{g}},$$

Instability regions for the swing



A half-cone in physical parameter space

Seyranian (2004)
Doklady Physics

$$(\beta/2)^2 + (2\omega/k - 1)^2 < r_k^2\varepsilon^2, \ \beta \geq 0$$

$$r_k = \frac{3}{4}\sqrt{a_k^2 + b_k^2} \quad a_k = \frac{1}{\pi}\int_0^{2\pi}\varphi(\tau)\cos k\tau d\tau, \quad b_k = \frac{1}{\pi}\int_0^{2\pi}\varphi(\tau)\sin k\tau d\tau,$$

Regular motion of the swing

Non-linear Mathieu-Hill equation for small angles θ

$$\ddot{q} + \beta\omega\dot{q} + [\omega^2 - \varepsilon(\ddot{\varphi} + \omega^2\varphi)]q - \frac{\omega^2}{6}q^3 = 0$$

$$\theta(\tau) = \frac{q(\tau)}{1 + \varepsilon\varphi(\tau)}$$

Three parameters: $\varepsilon, \beta, \omega$

Averaging method

$$\varphi(\tau) = \cos\tau$$

$$q(\tau) = A\cos(\tau/2 + \psi)$$

$$\frac{dA}{d\tau} = -\frac{\beta\omega}{2} - \frac{A\varepsilon(1-\omega^2)}{2}\sin(2\psi)$$

$$\frac{d\psi}{d\tau} = \omega - \frac{1}{2} - \frac{A^2\omega^2}{8} - \frac{\varepsilon(1-\omega^2)}{2}\cos(2\psi)$$

Periodic solutions

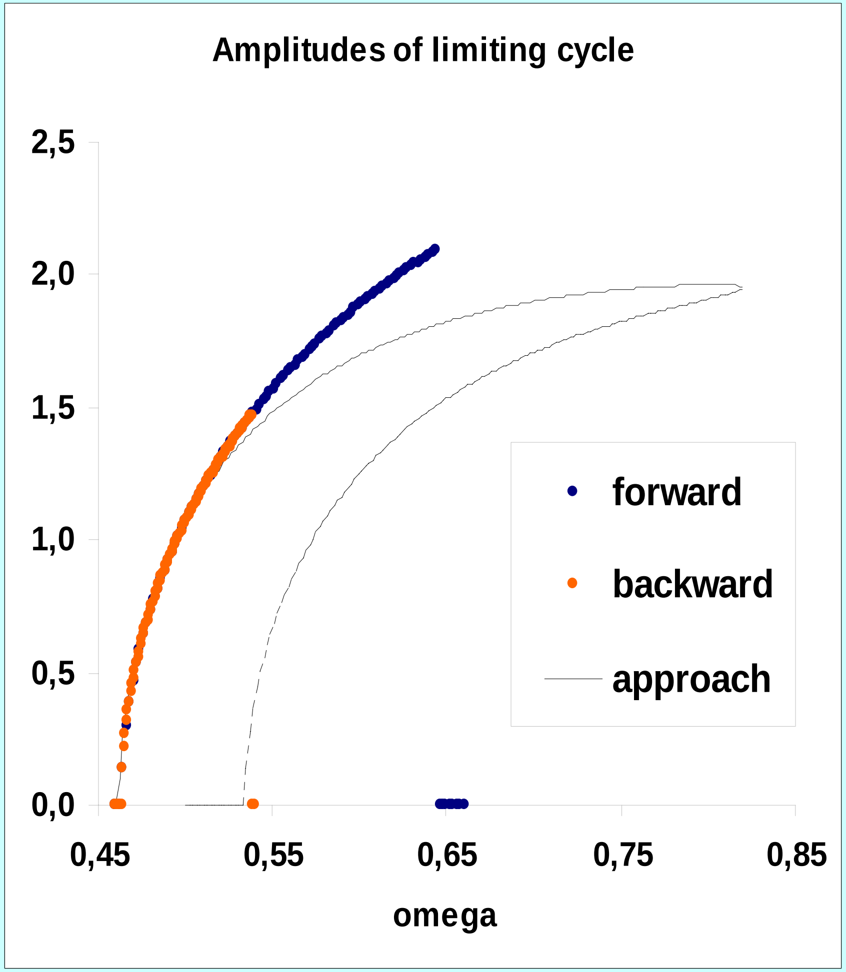
Steady motion: $\frac{dA}{d\tau} = 0\,, \quad \frac{d\psi}{d\tau} = 0$

Frequency-response curve

$$A^2 = \frac{4}{\omega^2}\left(2\omega - 1 \mp \sqrt{\varepsilon^2(1-\omega^2)^2 - \beta^2\omega^2}\right)$$

$$\psi = \frac{1}{2} Arctg\left(\mp \frac{4\beta\omega}{\sqrt{\varepsilon^2(1-\omega^2)^2 - \beta^2\omega^2}}\right) + \pi j\,,$$

$$j = 0, 1, 2, \ldots$$



Stability of periodic solutions

Linearization: $q(\tau) = A\cos(\tau/2 + \psi) + u(\tau), \qquad u(\tau) << 1$

Hill's equation with damping:

$$\ddot{u} + \beta\omega\dot{u} + (\omega^2 + \varepsilon\Phi(\tau))\,u = 0,$$

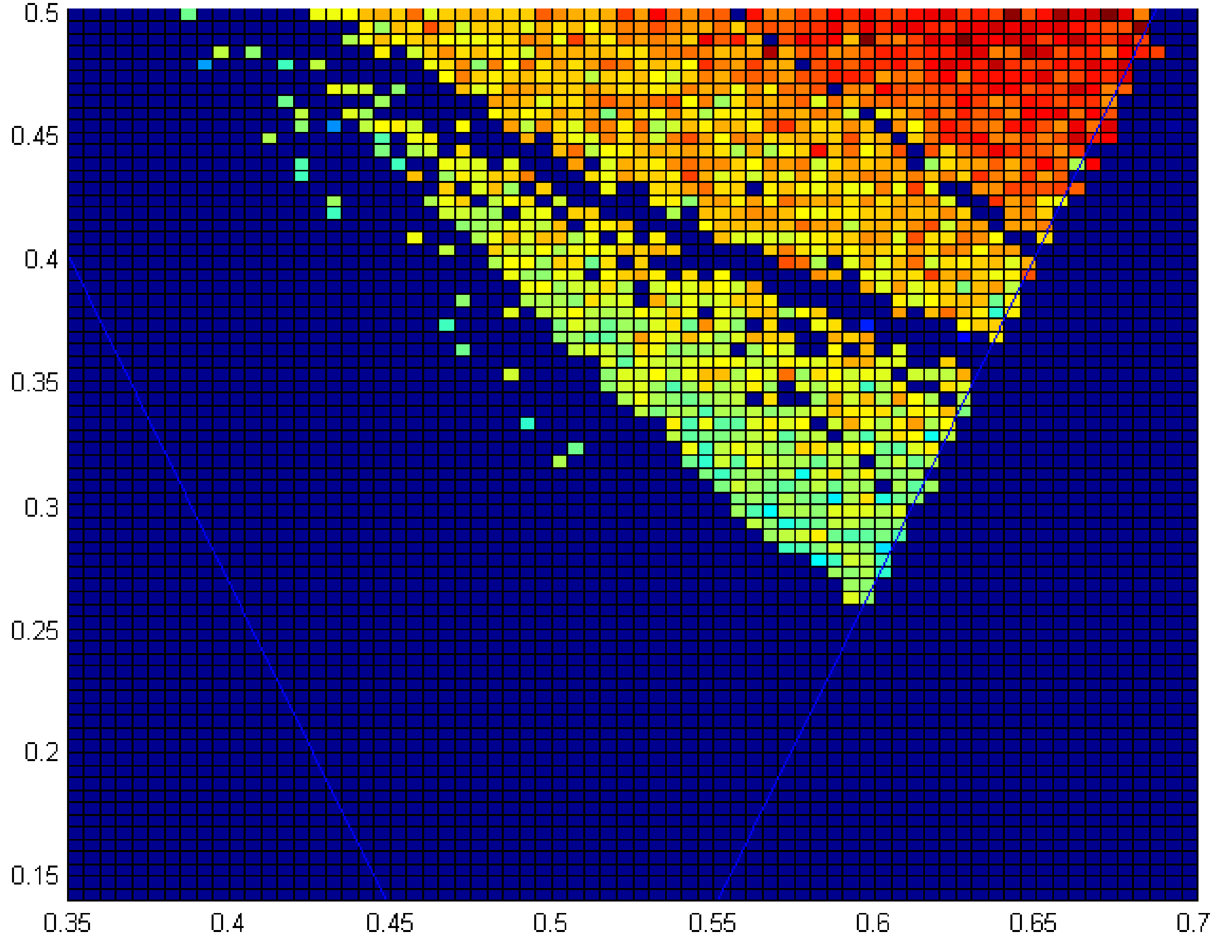
$$\Phi(\tau) = (1-\omega^2)\cos\tau - \frac{A^2\omega^2}{2\varepsilon}\cos^2\left(\frac{\tau}{2} + \psi\right)$$ *– periodic function*

Instability condition: Seyranian (2001). Doklady AN

$$A^2\omega^2\left(\frac{A^2\omega^2}{4} - 2\omega + 1\right) = \mp A^2\omega^2\sqrt{\varepsilon^2(1-\omega^2) - \beta^2\omega^2} < 0$$

Instability diagram and Lyapunov’s exponents

$\beta = 0.05$

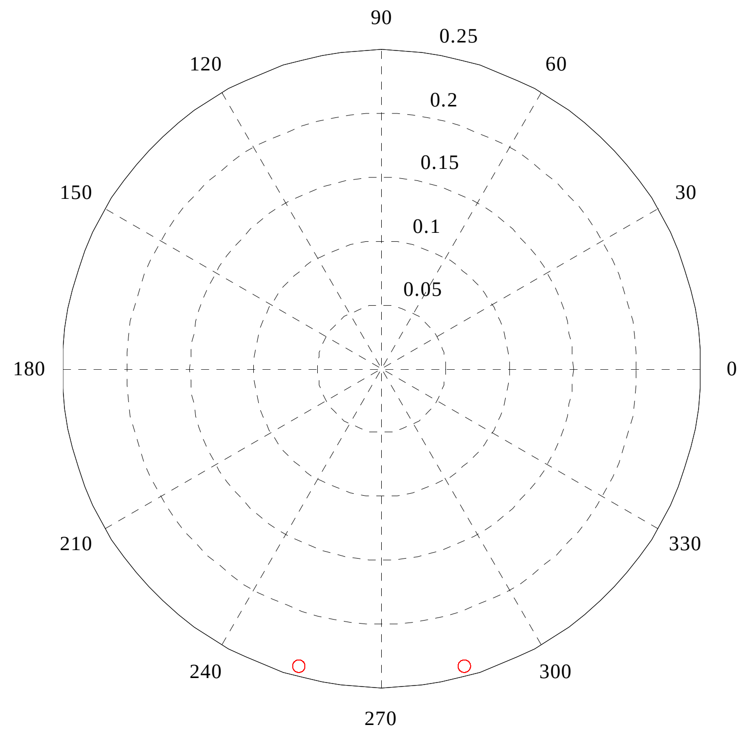


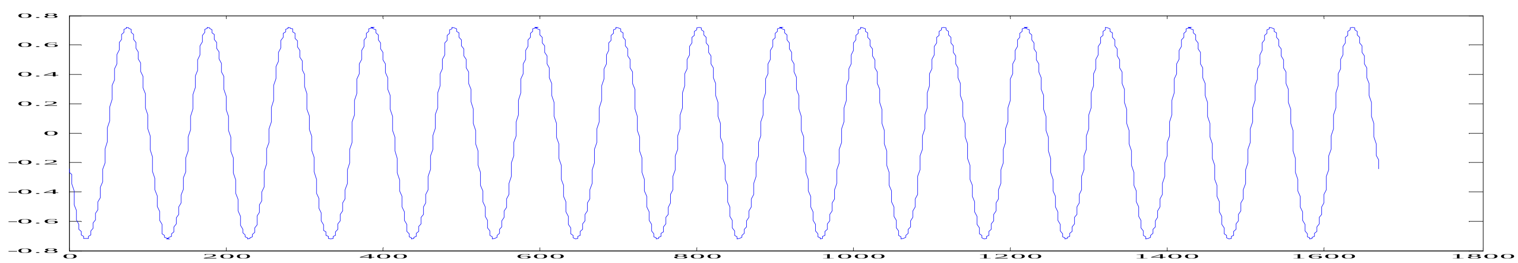
Numerical simulations

$\varepsilon = 0.3$ $\omega = 0.51$

– limiting cycle

with period **2**$(2\pi/\Omega)$





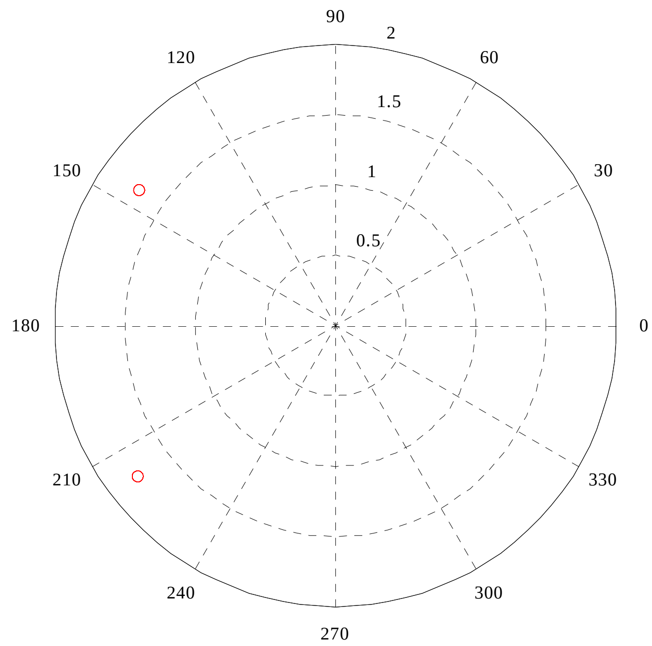
Numerical simulations

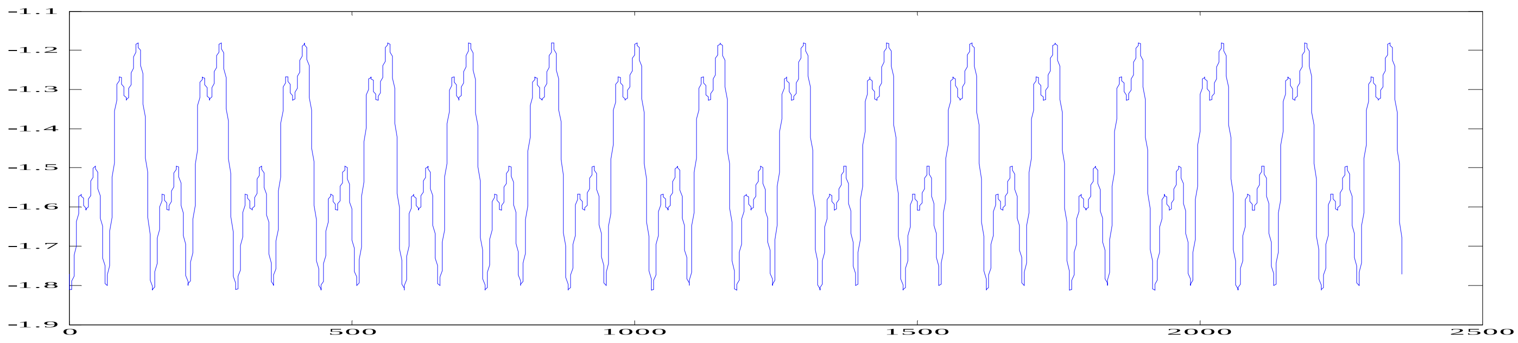
ε = 0.43 ω = 0.51

– rotation with

mean angular velocity **2**Ω

and period **2**$(2\pi/\Omega)$





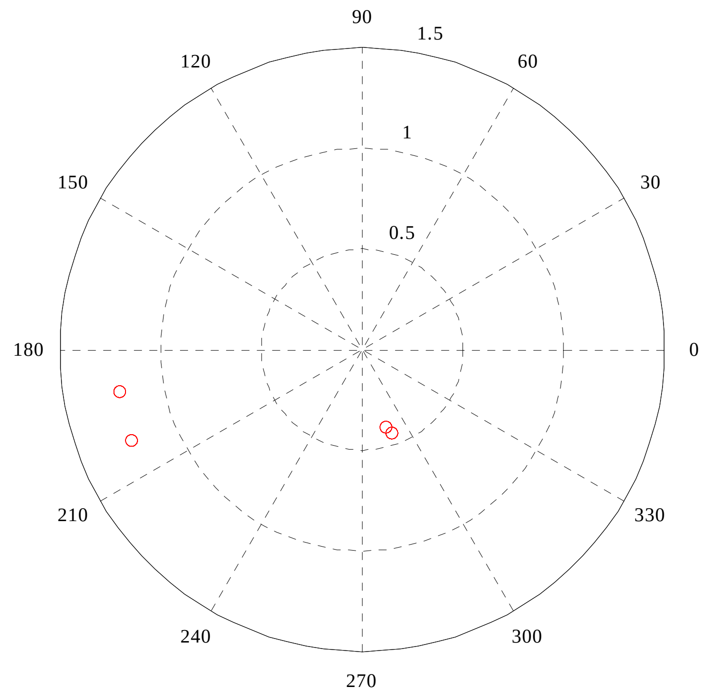
Numerical simulations

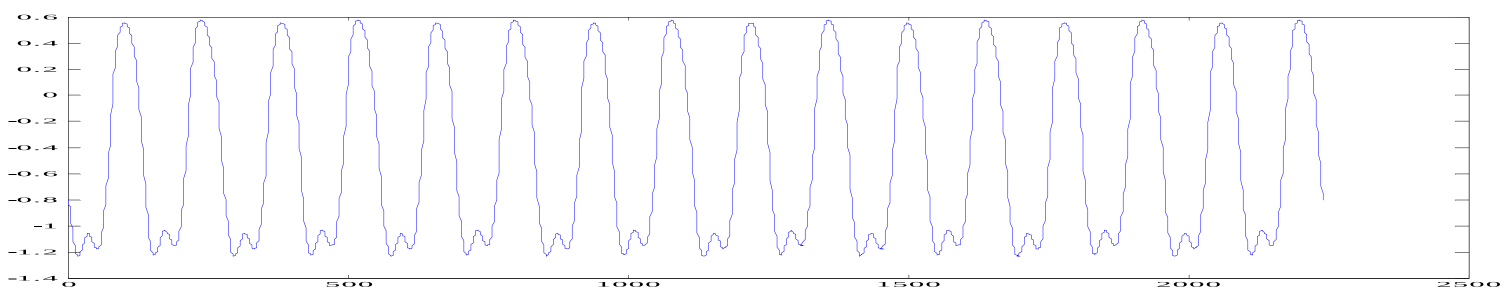
$\varepsilon = 0.45$ $\omega = 0.6$

– rotation with

mean angular velocity **3**Ω

and period **4**$(2\pi/\Omega)$





Numerical simulations

$\varepsilon = 0.5 \ \omega = 0.51$

CHAOS !!!

Poincare map

for the swing